Journal of Algebra **238**, 265–291 (2001)
doi:10.1006/jabr.2000.8629, available online at http://www.idealibrary.com on IDEAL®

Absolute Embeddings of Point–Line Geometries

A. Kasikova and E. Shult[1]

Department of Mathematics, Kansas State University, Manhattan, Kansas 66502

Received February 5, 2000

A method is given for showing that an embeddable point–line geometry possesses an absolutely universal projective embedding. This method is applied to show that virtually every embeddable Lie incidence geometry possesses an absolutely universal embedding. © 2001 Academic Press

1. INTRODUCTION

A *point–line geometry* is an incidence geometry $(\mathscr{P}, \mathscr{L})$ of points $(\mathscr{P})$ and lines $(\mathscr{L})$. We assume that every line is incident with at least two points and that distinct lines are incident with different sets of points—that is, they have distinct point–shadows.

Let V be a right vector space over a division ring k. The *projective space* $P(V)$ is the point–line geometry whose points and lines (called projective points and projective lines) are the 1- and 2-dimensional subspaces of V, under the natural incidence. A *projective embedding* of a point–line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ is an injective mapping e of the points into a spanning set of points of a projective space $\mathbf{P}(V)$, in such a manner that the image of the point–shadow of any line comprises all projective points of some projective line. Note that this induces an injection of the line set $\mathscr{L}$ into the line set of the projective space $\mathbf{P}(V)$. It is an *embedding over* k if V is a right vector space over k.

Given such an embedding $e: \Gamma \to \mathbf{P}(V)$ and any surjective semilinear transformation $t: V \to W$ with kernel K meeting the 2-space generated by each pair of embedded points trivially, the embedding can be carried onward to cosets of K to obtain an embedding $e': \Gamma \to \mathbf{P}(W)$. Specifically,

[1] Research supported by a grant from the National Science Foundation.

0021-8693/01 $35.00

for each point $p \in \mathscr{P}$, $e'(p) := t(e(p))$, a 1-space of W. In this case we say that the embedding e' is a *morphic image of* e or is *derived from* e. An embedding e is said to be *absolute* or *absolutely universal* if all embeddings e' of Γ are derived from e.

A few point–line geometries are known to possess absolutely universal embeddings. Classical projective spaces $\mathbf{P}(V)$ possess a unique projective embedding if $\dim V$ is at least three. Embeddable polar spaces of rank at least two [D, T, V] are classical and have an absolute embedding [T]. However, it has not been known in general whether the embeddable Lie incidence geometries even possess an absolute embedding.

Up to the present, there were only two known methods for showing that a Lie incidence geometry which is not already a projective space or a polar space possesses an absolute embedding: the method of Wells [W] and the method given in [S]. The former utilizes a lemma which is beautiful and simple but has limited application (mostly for spin and half-spin geometries). The latter method entails showing (1) that the embeddable geometry possesses Veldkamp lines and (2) that every geometric hyperplane arises from some fixed embedding e—a hypothesis which is sometimes difficult to verify (although this has recently been accomplished in [K] for the parapolar spaces of the long-root geometries).

The theorem below gives an inductive method for showing that an embeddable point–line geometry Γ has an absolute embedding. To facilitate a first reading of the theorem, a *subspace* of a point–line geometry $(\mathscr{P}, \mathscr{L})$ is a subset $\mathscr{S}$ of $\mathscr{P}$ such that for any line $L \in \mathscr{L}$, its *point–shadow*, the set $sh(L)$ of all points incident with L, is either contained in S or meets S in at most one point.

THEOREM 1 (main theorem). *Suppose $\mathscr{R}$ is a family of subspaces of a point–line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ having these properties*:

1. *Every subspace $R \in \mathscr{R}$ is connected, contains a line, and possesses an absolute embedding over k.*

2. *Every line lies in a member of $\mathscr{R}$.*

3. *Let $\Sigma = (\mathscr{R}, \sim)$ be one of the following graphs whose vertex set is $\mathscr{R}$. Case* 1: *two distinct members of $\mathscr{R}$ are adjacent if and only if their intersection is connected and contains a line, and Case* 2: *if no intersection of distinct members of $\mathscr{R}$ contains a line, then adjacent pairs are those which meet at a point.*

(a) *In Case* 1, *assume*

(E1) *for any edge (A, B) in Σ—that is, $A \cap B$ contains a line—then $A \cap B$ is a connected subspace, and*

(Σ_p) *for each point p, the subgraph Σ_p induced on the vertex set $\{R \in \mathscr{R} \mid p \in R\}$ is connected.*

(b) *In Case* 2, *we assume the following*:

(E2) *If* $A \cap B = \{p\}$, *a point—that is*, (A, B) *is an edge of* Σ*—then there exists a subspace* Y *with an absolute embedding over* k, *which contains* p *and whose intersections* $A \cap Y$ *and* $B \cap Y$ *each contain a line on* p.

4. *If* A, B, *and* C *are members of* $\mathscr{R}$ *forming a triangle in* Σ, *the global intersection* $A \cap B \cap C$ *is either*

(T1) *non-empty*, *or*

(T2) *is empty*, *and there exists a connected subspace* X *having an absolute embedding over* k *which meets each of* A, B, *and* C *at connected subspaces and which meets each of the three pairwise intersections among* A, B, *and* C, *non-trivially*.

5. *Finally we assume that the graph* Σ *is simply connected*.

Then, *if* Γ *is embeddable over* k, *it has an absolute embedding over* k.

Section 2 contains several subsections devoted to basic concepts. The first gives a brief explanation of graph morphism and simple connectedness and introduces morphisms of point–line geometries. The next gives an account of the category of embeddings over k, universal hulls, and absolute embeddings. The last introduces the notion of a point–line presheaf, a key ingredient in the proof of the main theorem which occupies Section 3. In Section 4, the theorem is applied to show that every embeddable Lie incidence geometry of sufficient rank possesses an absolute embedding.

2. BASICS

2.1. *Simple Connectedness and Morphisms of Point–Line Geometries*

Let $G = (V, E)$ and $G' = (V', E')$ be simple graphs. A *morphism of graphs* ν: $G \to G'$ is a mapping ν: $V \to V'$ of vertex sets such that for every edge $e = (x, y) \in E$, either $\nu(x) = \nu(y)$ or $(\nu(x), \nu(y))$ is an edge (called $\nu(e)$) in E'. The set of preimages of an image vertex $z = \nu(x)$ is called the *fiber above* z. Let $\Delta_H(x)$ be the collection of all vertices adjacent to the vertex x in the graph H, where we take H to be either G or G'. A morphism f: $G \to G'$ is called a *fibering* if and only if it is locally bijective —that is, for each vertex x of G, the restriction of f to $\Delta_G(x)$ induces a bijection $\Delta_G(x) \to \Delta_{G'}(f(x))$. Several standard phenomena arise when f is a fibering: (1) all fibers are cocliques (graphs without edges), (2) paths in G' beginning at an image point $f(x)$ have a unique lift to a path of G' beginning at x, and (3) f is vertex-surjective if G' is connected and V is non-empty.

Let $\mathscr{C}$ be any collection of circuits of G'. A fibering $f: G \to G'$ is said to be a *$\mathscr{C}$-covering* if and only if (1) f is surjective on vertices, (2) the domain graph G is connected, and (3) any circuit in $\mathscr{C}$ lifts to a (pointed) circuit (as walked from the preimage of any of its vertices) of G. A $\mathscr{C}$-covering $\kappa: U \to G'$ is said to be a *universal $\mathscr{C}$-covering* if and only if for any other $\mathscr{C}$-covering $f: G \to G'$ there is a graph morphism $\phi: U \to G$ such that $\kappa = f \circ \phi$. Any two universal $\mathscr{C}$-covers are isomorphic as graphs. It is a result of the folklore that universal $\mathscr{C}$-covers always exist. For a full account of this using $\mathscr{C}$-homotopy classes of paths, see the book by Aschbacher [A] (where the theory is derived from Aschbacher and Segev [AS]) or the class notes in [S1].

Let $\mathscr{T}$ be the class of circuits of length 3 in the graph G' ("$\mathscr{T}$" stands for "triangles"). In any $\mathscr{T}$-covering $f: G \to G'$, the local induced mappings $\Delta_G(x) \to \Delta_{G'}(f(x))$ are all graph isomorphisms. A graph G' is said to be *simply connected* if and only if every $\mathscr{T}$-covering $G \to G'$ is an isomorphism. Equivalently, a graph is simply connected if and only if the identity mapping is a universal $\mathscr{T}$-covering.

A *walk* in a graph $G = (V, E)$ is a sequence of vertices $(x_0, x_1, \dots, x_n)$ such that $(x_i, x_{i+1}) \in E$, $i = 0, \dots, n-1$. This implies $x_i \neq x_{i+1}$, but does not *a fortiori* prevent other identities among the x_i. In a graph, $G = (V, E)$, an *elementary $\mathscr{T}$-homotopy* is the replacement of a walk $p = (x_0, x_1, \dots, x_n)$ by a walk $p' = (x_0, \dots, x_i, a, x_j, x_{j+1}, \dots, x_n)$ where $j = i$ or $i + 1$ or the reverse replacement. This process does not disturb the initial and terminal vertices of the walk. Two walks p and q are $\mathscr{T}$-homotopic if one can be derived from the other by a sequence of $\mathscr{T}$-homotopies. A printed circuit $(x_0, \dots, x_n = x_0)$ is said to be *contractible* if and only if it is $\mathscr{T}$-homotopic to the trivial circuit (x_0) of length zero. (The word "pointed" is here only to indicate that there is a specified beginning and end for a walk through the circuit.)

LEMMA 2. *A graph is simply connected if and only if all of its pointed circuits are contractible.*

We present two theorems which can be used to establish simple connectivity of a graph.

LEMMA 3 (Weetman [WT]). *Suppose G is a graph. For any vertex x, let $\Delta_i(x)$ denote the induced subgraph on the set of vertices whose distance from vertex x is exactly i. For each $i > 1$ and vertex x suppose the following*:

(W1) *For each vertex a in $\Delta_i(x)$, the induced subgraph on $\Delta_1(a) \cap \Delta_{i-1}(x)$ is connected.*

(W2) *For each edge $e = (a, b)$ of the subgraph $\Delta_i(x)$, $\Delta_1(a) \cap \Delta_1(b) \cap \Delta_{i-1}(x)$ is non-empty.*

Then Γ *is simply connected.*

Proof. The hypothesis makes all circuits contractible (see [S1]).

LEMMA 4 (Pasini [P]). *Let* Γ *be a diagram geometry over the typeset* I *of rank at least three, and fix a type* $i \in I$. *Let* $\mathscr{P}$ *be the objects of type* i *in* Γ *and let* $\mathscr{L}$ *be the set of all flags whose typeset is the minimal set* $J(i)$ *separating* i *from the rest of the nodes in the basic diagram for* Γ. *Let* $G = (\mathscr{P}, \sim)$ *be the point-collinearity graph of* $(\mathscr{P}, \mathscr{L})$ *and let* $\mathscr{K}$ *be the system of subgraphs of* G *produced by the cliques formed by the point–shadows of lines together with the subgraphs of* G *induced on the i-shadows of all other objects of type* $j \in I - (\{i\} \cup J(i))$. *Let* $\mathscr{C}(\mathscr{K})$ *be the collection of all circuits of* G *which are circuits of one of the subgraphs of* $\mathscr{K}$.

Suppose that

1. $(\mathscr{P}, \mathscr{L})$ *is a partial linear space, and*
2. Γ, *considered as a multipartite graph, is simply connected.*

Then G *is simply* $\mathscr{C}(\mathscr{K})$*-connected: that is, circuits of the graph* G *are* $\mathscr{C}(\mathscr{K})$*-contractible to a circuit of* $\mathscr{C}(\mathscr{K})$. *In particular, if each subgraph of* $\mathscr{K}$ *is simply connected—that is,* $\mathscr{T}$*-simply connected—then the graph* G *is itself simply connected.*

Remark. Lemma 4 is a special case of Theorem 12.64, Chap. 12, p. 398 of Pasini [P], which asserts the equality of two fundamental groups. This is an extremely useful theorem in contexts beyond simple connectivity of a graph via triangles as it is used here. Note that the lemma itself can be iterated to build up the simple connectivity of the subgraphs in $\mathscr{K}$. For example the simple connectedness of the point-collinearity graph of $E_{8,8}$ follows from that of $E_{7,7}$ and its convex subgraphs.

Remark. We have presented these two lemmas (both of 1994 vintage) since each has advantages and disadvantages. Weetman's theorem has an extremely simple proof, but requires special information about the local graph far from a point. This information is not so accessible in a high-diameter graph like that derived from $E_{8,8}$, which is formed from an association scheme of 35 classes. Lemma 4 is much more sophisticated than Weetman's lemma, and it depends ultimately on the fact that chamber systems of rank-three buildings are simply 2-connected. However, its versatility is tremendous: all of the graphs listed in the corollary below can be proved to be simply connected by this lemma. (For an account of Pasini's theorem rendered as an application of coverings of graphs, the reader is invited to consult the last section of [S1].)

COROLLARY 5. *The following graphs are simply connected*:

1. *complete graphs*

2. *the point-collinearity graphs of the following Lie incidence geometries*:

(a) *Any non-degenerate polar space of rank at least three.*

(b) *The point-collinearity graphs of the half-spin geometries* $D_{n,n}(k)$, *where* k *is a field.*

(c) *The strong parapolar spaces* $E_{6,1}$ *and* $E_{7,1}$.

(d) *The long-root geometries* $F_{4,1}$, $E_{6,4}$, $E_{7,7}$, *and* $E_{8,1}$.

(e) *The Lie incidence geometry* $E_{8,8}$.

Proof. Any student can verify the Weetman conditions in all but the last two cases. Here, the very general Pasini lemma provides an easy argument that can be made to cover all the cases listed just as well. Lemma 4 reduces the $\mathscr{T}$-contractibility to circuits within the induced subgraphs determined by the point–shadows of lines and all other objects of the geometry. Since the latter are point-collinearity graphs of Lie incidence geometries drawn from the same conclusion list, but of lower geometric rank, one has an induction principle at work.

2.2. *Embeddings*

Now let $\Gamma = (\mathscr{P}, \mathscr{L})$ and $\Gamma' = (\mathscr{P}', \mathscr{L}')$ be point–line geometries. A *morphism of point–line geometries* f: $\Gamma \to \Gamma'$ is a mapping f: $\mathscr{P} \to \mathscr{P}'$ for which the image of a point–shadow of any line of $\mathscr{L}$ is the full point–shadow of a line of $\mathscr{L}'$. If f is injective on vertices, it is called an *embedding of* Γ *into* Γ'. Obviously, any morphism of point–line geometries induces a corresponding graph morphism of the point-collinearity graphs of the geometries.

For any right vector space V over a division ring k, let $\mathbf{P}(V)$ denote the projective space over V—that is, the point–line geometry whose points and lines are the 1- and 2-subspaces of V under the usual incidence. For every vector subspace K of V, let $A(V; K)$ be the *attenuated space of* V *over* K which is the subgeometry of $P(V)$ whose points and lines are the 1- and 2-subspaces of V which intersect K trivially. There is always a natural morphism of point–line geometries

$$\rho_K : A(V; K) \to \mathbf{P}(V/K),$$

taking each point p of $A(V; K)$ to $p + K$ and each line L of $A(V; K)$ to $L + K$, a 2-subspace of V/K. Similarly, any surjective semilinear transformation t of V with kernel K induces a morphism

$$\tau_K : A(V; K) \to \mathbf{P}(V/K).$$

Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a point–line geometry. An *embedding of* Γ *over* k is an injective mapping e: $\mathscr{P} \to$ 1-subspaces of V, where V is a right vector space over k, such that

1. For each line L of Γ, $\{e(p) \mid p * L\}$ is the set of all 1-subspaces of a 2-subspace $e(L)$ of V.
2. $e(\mathscr{P})$ spans V.

Such an embedding is denoted by the symbol e: $\Gamma \to \mathbf{P}(V)$. A point–line geometry Γ is said to be *embeddable over* k if and only if there exists an embedding over k. It is just called *embeddable* if it is embeddable over k for some division ring k.

It may happen that for some embedding e: $\Gamma \to \mathbf{P}(V)$, the image subgeometry $(e(\mathscr{P}), e(\mathscr{L}))$ actually sits in some attenuated subgeometry $A(V, K)$ (this happens only if $x, y \in \mathscr{P}$ implies $\langle e(x), e(y)\rangle_{\mathbf{P}(V)} \cap K = \{0\}$). In that case e can be composed with a morphism τ_K: $A(V; K) \to \mathbf{P}(V/K)$, derived from some surjective semilinear transformation t: $V \to V/K$ whose kernel is K. The result is a new embedding $e' := \tau_K \circ e$: $\Gamma \to \mathbf{P}(V/K)$. We call this transformation from e to e' via τ_K a *morphism of embeddings* and denote it as τ: $e \to e'$. It is called an *equivalence* if $K = 0$, in which case τ is invertible. Clearly all morphisms among embeddings over k can always be composed when there is a common middle term.

The embeddings of Γ over k form a category $\mathscr{E}$ with the morphisms as described in the previous paragraph. For a given embedding e: $\Gamma \to \mathbf{P}(V)$ over k, let $\mathscr{E}_e$ be the induced subcategory of all embeddings f for which there is a morphism $f \to e$. ("Induced" means that we inherit all preexisting morphisms between embeddings in $\mathscr{E}_e$). Any initial object in $\mathscr{E}_e$, if it exists, is unique up to equivalence and is called the *universal hull of* e. Thus the universal hull of e is an embedding $\hat{e}$ together with a morphism κ: $\hat{e} \to e$ such that for any morphism ϕ: $f \to e$, there is a morphism λ_ϕ: $\hat{e} \to f$ such that $\kappa = \phi \circ \lambda_\phi$. It is a very important theorem of Ronan [R], that for any embedding e, of Γ, the relatively universal hull $\hat{e}$ exists.

An initial object with respect to the full category $\mathscr{E}$, if it exists, is also unique up to equivalence and is called an *absolutely universal embedding* or just *absolute embedding* for short. If u is an absolute embedding over k, every embedding of Γ over k is derived from u by a morphism.

As an example, it is a consequence of the classification of polar spaces and a theorem of Dienst [D], that any *embeddable* polar space of rank at least two either possesses an absolute embedding over some division ring k or, in the rank two case, the relatively absolute embeddings can be parameterized by derivations of k.

2.3. *Sheaves and Embeddings*

Let $\Gamma = (\mathscr{P}, \mathscr{L})$ be a point–line geometry. Let k be any division ring. The phrase "d-space" in this section will always refer to a right vector space of dimension d over k. We say that $\mathscr{F}$ is a *presheaf of* Γ *over* k if and only if, for each point p, line L, and flag pL, we have a 1-space $\mathscr{F}_p$, a 2-space $\mathscr{F}_L$, and a 1-space $\mathscr{F}_{pL}$, and, for each flag pL, k-semilinear "connecting mappings"

$$\mathscr{F}_p \xleftarrow{\phi_{Lp}} \mathscr{F}_{pL} \xrightarrow{\phi_{pL}} \mathscr{F}_L,$$

such that if

$$i_{pL} := \phi_{pL} \circ \phi_{Lp}^{-1} : \mathscr{F}_p \to \mathscr{F}_L,$$

the mapping $p \to i_{pL}(\mathscr{F}_p)$, induces a bijection

$$\{ p \mid p * L \} \to \text{1-subspaces of } \mathscr{F}_L.$$

Suppose V is a right vector space over the division ring k. Any embedding $e: \Gamma \to \mathbf{P}(V)$ defines a presheaf $\mathscr{F}^e$ over k where $\mathscr{F}_p = \mathscr{F}_{pL} = e(p)$, $\mathscr{F}_L = e(L)$, and the connecting mappings ϕ_{Lp} and ϕ_{pL} are, respectively, the identity and the inclusion mappings.

Suppose $\mathscr{F}$ and $\mathscr{F}'$ are presheaves over k for a geometry Γ. An *isomorphism* $f: \mathscr{F} \to \mathscr{F}'$ is a collection of k-semilinear mappings

$$f_p : \mathscr{F}_p \to \mathscr{F}'_p,$$
$$f_L : \mathscr{F}_L \to \mathscr{F}'_L,$$
$$f_{pL} : \mathscr{F}_{pL} \to \mathscr{F}'_{pL},$$

all associated with the same automorphism of k, which "commute" with the respective connecting mappings ϕ and ϕ' of the two presheaves—specifically,

$$\phi'_{Lp} \circ f_{pL} = f_p \circ \phi_{Lp}, \qquad \text{and}$$
$$\phi'_{pL} \circ f_{pL} = f_L \circ \phi_{pL}.$$

A very important example of an isomorphism is scalar multiplication of a presheaf. Fix a presheaf $\mathscr{F}$ over k of a fixed geometry Γ having connecting morphisms ϕ_{pL} and ϕ_{Lp} at each flag pL of Γ. Fix a non-zero scalar α from the division ring k. The presheaf $\mathscr{F}\alpha$ has the same terms as $\mathscr{F}$—that, is $(\mathscr{F}\alpha)_X = \mathscr{F}_X$ for $X = p$, L or pL—but the connecting morphism $\phi_{pL}: \mathscr{F}_{pL} \to \mathscr{F}_L$ has been replaced by $\phi_{pL}\alpha$; that is, scalar multiplication of $\mathscr{F}_{pL}$ by α composed with ϕ_{pL}, while the morphism ϕ_{Lp} toward points remains the same. Then there is an isomorphism of presheaves

$$f\colon \mathscr{F}\alpha \to \mathscr{F}$$

given by setting $f_L := 1_{\mathscr{F}_L}$ and letting f_p and f_{pL} be scalar multiplication by α on the subspaces $\mathscr{F}_p$ and $\mathscr{F}_{pL}$, respectively. (Note that in $\mathscr{F}\alpha$, the mapping $i_{pL}\colon \mathscr{F}_p \to \mathscr{F}_L$ in $\mathscr{F}$ has also been altered by a scalar multiplication.) We say that the presheaf $\mathscr{F}\alpha$ is *proportional to* $\mathscr{F}$ or is *a scalar multiple of* $\mathscr{F}$.

Let $C_1 := \oplus \mathscr{F}_{pL}$ (summed over all flags pL of Γ) and set $C_0 := \oplus \mathscr{F}_p \oplus \mathscr{F}_L$ (summed independently over the points and lines of Γ). There is a unique k-linear "boundary transformation" $\partial\colon C_1 \to C_0$ for which

$$\partial(v) = \phi_{Lp}(v) - \phi_{pL}(v), \qquad v \in \mathscr{F}_{pL},$$

where pL is any flag. Then for the natural quotient mapping

$$\phi_{\mathscr{F}}\colon C_0 \to C_0/\partial(C_1) := H_0(\mathscr{F}),$$

one sees that $\dim \phi_{\mathscr{F}}(\mathscr{F}_p) \le 1$ and $\dim \phi_{\mathscr{F}}(\mathscr{F}_L) \le 2$ for any point p and line L.

The following proposition is proved in Ronan's fundamental paper [R]:

THEOREM 6. *Suppose $\mathscr{F}$ is a presheaf over k of the point–line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$.*

1. *Suppose for any two distinct points p and q of $\mathscr{P}$, $\langle \mathscr{F}_p, \mathscr{F}_q \rangle_{C_0} \cap \partial(C_1) = 0$.*

(a) *Then the mapping e which takes each point p to the coset space $\mathscr{F}_p + \partial(C_1)$ defines an embedding*

$$e(\mathscr{F})\colon \Gamma \to \mathbf{P}(H_0(\mathscr{F})).$$

(b) *If $\mathscr{F}$ is isomorphic to $\mathscr{F}^{e'}$ for some embedding $e'\colon \Gamma \to \mathbf{P}(V)$, then there is a semilinear surjection $\pi\colon H_0(\mathscr{F}^{e'}) \to V$ inducing a morphism of embeddings*

$$e(\mathscr{F}^{e'}) \to e'.$$

(c) *Under the hypothesis of* (b), *$e(\mathscr{F}^{e'})$ is the relatively universal hull of e'.*

2. *If Γ is embeddable over k with presheaf $\mathscr{F}$, then the embedding of part* 1 *exists.*

Proof. Part 1. All three statements follow from Propositions 1–3 of [R], except that the hypothesis of his Proposition 1 has been slightly strengthened to ensure that $e(\mathscr{F})$ is point-injective.

Part 2. We need only show that for two distinct points, p and q, the canonical injections of $\mathscr{F}_p$ and $\mathscr{F}_q$ in C_0 cannot be congruent modulo $\partial(C_1)$. But by hypothesis, there exists an embedding e with presheaf $\mathscr{F}$, and from Part 1, for every point $x \in \mathscr{P}$, $e(x) = \pi(\mathscr{F}_x + \partial(C_1))$, where π is the semilinear transformation of (b) of Part 1 of the statement of the theorem. Since e is point-injective, $e(p) \neq e(q)$ and so we could not have had $\mathscr{F}_q + \partial(C_1) = \mathscr{F}_p + \partial(C_1)$. Thus the hypotheses of Part 1 hold.

It will be useful to us to recast the notion of presheaf in the following equivalent way: a *point–line presheaf* $\mathscr{F} = (\mathscr{F}, \mathscr{I})$ *of a geometry* $\Gamma = (\mathscr{P}, \mathscr{L})$ *over* k is a collection of 1-spaces and 2-space $\{\mathscr{F}_p\}$ and $\{\mathscr{F}_L\}$ indexed by $\mathscr{P}$ and $\mathscr{L}$, respectively, and a collection $\mathscr{I}$ of injective k-linear mappings i_{pL}: $\mathscr{F}_p \to \mathscr{F}_L$, indexed by the flags pL, having the property that for each line L, the mapping $p \to i_{pL}(\mathscr{F}_p)$ induces a bijection

$$\text{point–shadow of } L := \{p \mid p * L\} \to \text{1-subspaces of } \mathscr{F}_L.$$

For every point–line presheaf $\mathscr{F}$ (in this sense), one obtains a presheaf $\sigma\mathscr{F}$ in the previous sense by setting $\mathscr{F}_{pL} = \mathscr{F}_p$, letting ϕ_{Lp} be the identity mapping on this space, and letting $\phi_{pL} := i_{pL}$. Conversely, for any presheaf, $\mathscr{F}$, one obtains a point–line presheaf $\gamma\mathscr{F} = (\mathscr{F}', \mathscr{I})$, where $\mathscr{F}'$ is the collection of k-spaces $\{\mathscr{F}_p\} \cup \{\mathscr{F}_L\}$ and $\mathscr{I}$ is the collection of k-linear mappings

$$\left\{i_{pL} := \phi_{pL} \circ \phi_{Lp}^{-1} \mid pL \text{ a flag}\right\}.$$

Note that $\gamma\sigma\mathscr{F} = \mathscr{F}$ for any point–line presheaf $\mathscr{F}$.

Now an isomorphism of point–line presheaves, f: $\mathscr{F} \to \mathscr{F}'$, consists of two collections of k-semilinear bijections

$$\{f_p : \mathscr{F}_p \to \mathscr{F}'_p \mid p \in \mathscr{P}\} \qquad \text{and}$$

$$\{f_L : \mathscr{F}_L \to \mathscr{F}'_L \mid L \in \mathscr{L}\},$$

such that for each flag pL

$$f_L \circ i_{pL} = i'_{pL} \circ f_p.$$

(There is only one commutative diagram of interest in this case. Note that the "functors" γ and σ take isomorphisms in their category to specific induced isomorphisms among sheaves in their image categories.)

Obviously any embedding $e\colon \Gamma \to \mathbf{P}(V)$ defines a point–line presheaf $\mathscr{F}^e$, obtained by setting $\mathscr{F}_p^e = e(p)$ and $\mathscr{F}^e(L) = e(L)$ and letting i_{pL} be the containment mapping $e(p) \hookrightarrow e(L)$, for each point p, line L, and flag pL, respectively.

If $\mathscr{F} = (\mathscr{F}, \mathscr{I})$ is a point–line presheaf, the scalar multiple $\mathscr{F}\alpha$ is the point–line presheaf $(\mathscr{F}, \mathscr{I} \circ \alpha)$ in which each mapping i_{pL} has been precomposed by scalar multiplication of $\mathscr{F}_p$ by a constant α. There is an isomorphism of point–line presheaves: $\mathscr{F}\alpha \to \mathscr{F}$, in which f_p is scalar multiplication of $\mathscr{F}_p$ by α and f_L is the identity mapping on $\mathscr{F}_L$.

Fix a point–line sheaf $\mathscr{F}$. Let $C_0 = \oplus \mathscr{F}_p \oplus \mathscr{F}_L$ as for presheaves, and let $\eta_p\colon \mathscr{F}_p \to C_0$ and $\eta_L\colon \mathscr{F}_L \to C_0$ be the canonical injections into the direct sum. Let K be the subspace of C_0 spanned by $\{\eta_p(v) - \eta_L(i_{pL}(v)) \mid v \in \mathscr{F}_p, pL \text{ a flag}\}$. Then $H_0(\sigma\mathscr{F}) = C_0/K$ affords an embedding if, for any two distinct points, p and q, $\dim(\langle \eta_p(\mathscr{F}_p), \eta_q(\mathscr{F}_q)\rangle + K)$ in C_0/K is always 2. When this occurs, this embedding is its own universal hull.

We conclude this section with a warning: The existence of a presheaf does not guarantee that an embedding exists.

3. PROOF OF THE MAIN THEOREM

Assume the hypotheses concerning the family $\mathscr{R}$. It suffices to show that any two embeddings over k of the geometry Γ yield isomorphic point–line presheaves, for in that case, both embeddings possess the same universal hull. It then follows that all embeddings over k are derived from that hull, and so the latter is an absolute embedding, proving the theorem.

So suppose that $e_i\colon \Gamma \to \mathbf{P}(V_i)$, $i = 1, 2$, are any two arbitrary embeddings of the geometry Γ, where the V_i are right vector spaces over the division ring k. With each embedding e_i, $i = 1, 2$, there is defined a point–line presheaf $\mathscr{F}^{e_i}$ which we write as $\mathscr{F}^{(i)}$ in order to avoid subscripts of superscripts. Thus $\mathscr{F}_p^{(i)} = e_i(p)$ and $\mathscr{F}_L^{(i)} = e_i(L)$ for every point p and line L.

Now for every subspace $R \in \mathscr{R}$, we obtain by restriction embeddings $e_i|_R\colon\ R \to \langle e_i(R)\rangle_{V_i}$, $i = 1, 2$, for the subspace R, each providing a point–line presheaf $\mathscr{F}^{(i)}(R)$ of the point–line geometry of the subspace R. (Note that $\mathscr{F}^{(i)}(R)$ is nothing more than the point–line presheaf $\mathscr{F}^{(i)}$ with its points and lines and connecting morphisms restricted to points and lines of R and the containment maps of their images.)

Now since R possesses an absolute embedding over k, the two point–line presheaves for R, $\mathscr{F}^{(1)}(R)$ and $\mathscr{F}^{(2)}(R)$, are isomorphic. This means that the isomorphism

$$t_R\colon \mathscr{F}^{(1)}(R) \to \mathscr{F}^{(2)}(R)$$

is described by two collections of k-semilinear transformations, all associated with the same automorphism,

$$\left\{t_{R,p}\colon \mathscr{F}^{(1)}(R)_p = e_1(p) \to \mathscr{F}^{(2)}(R)_p = e_2(p)\right\}$$
$$\left\{t_{R,L}\colon \mathscr{F}^{(1)}(R)_L = e_1(L) \to \mathscr{F}^{(2)}(R)_L = e_2(L)\right\}$$

indexed by the points and lines of R. Since the connecting mappings for each presheaf is a inclusion mapping, the required commutative diagram amounts to the assertion that for any flag pL of R,

(*) the restriction of the semilinear mapping $t_{R,L}\colon e_1(L) \to e_2(L)$ to the subspace $e_1(p)$ reproduces the mapping $t_{R,p}\colon e_1(p) \to e_2(p)$.

But isomorphism of point–line presheaves $t_R\colon \mathscr{F}^{(1)}(R) \to \mathscr{F}^{(2)}(R)$ is not unique. Suppose that $t'_R\colon \mathscr{F}^{(1)}(R) \to \mathscr{F}^{(2)}(R)$ is a second isomorphism. Then for any line L of R, condition (*) implies that $t_{R,L}$ and $t'_{R,L}$ agree in their assignment of 1-subspaces of $e_1(L)$, and so differ only by precomposition by scalar multiplication—that is, $t'_{R,L} = t_{R,L} \circ \alpha_L$ where α_L indicates multiplication of the 2-space $e_1(L)$ by a non-zero scalar α of k. Then by (*), for each point p of L, $t'_{R,p} = t_{R,p} \circ \alpha_p$ where similarly α_p is (right) scalar multiplication of all vectors of $e_1(p)$ by α. Now, if N is any second line of R on point p, we must have $t'_{R,N} := t_{R,N} \circ \alpha_N$ in order to maintain (*) for the flag pN of R. Since R is a connected subspace, this scalar α is perpetuated throughout R and we may say that $t'_R = t_R\alpha$ where the term on the right is the isomorphism of point–line presheaves whose indexed semilinear mappings are those of t_R, each precomposed with a prior scalar multiplication of their domain spaces by α. For natural reasons we say that t'_R is a *scalar multiple of the isomorphism* t_R by α. What we are saying, then, is this:

(Step 1) Since R is connected and contains lines, the isomorphism of point–line presheaves $t_R\colon \mathscr{F}^{(1)}(R) \to \mathscr{F}^{(2)}(R)$ is determined up to scalar multiplication.

Now suppose we are in Case 1; S is a second member of $\mathscr{R}$ so that (R, S) is an edge of Σ. Then $R \cap S$ is a connected subspace which contains a line L (condition (E1)). Then (with the notation of the previous paragraph) (*) implies that $t_{R,L}$ and $t_{S,L}$ produce the same disposition of 1-spaces of $e_1(L)$ (each takes $e_1(p)$ to $e_2(p)$) and so $t_{R,L}$ is a scalar multiple of $t_{S,L}$. But since $R \cap S$ is connected, this same scalar multiple of t_R agrees with t_S when restricted to all of $e_1(R \cap S)$.

In Case 2, if $R \cap S$ is a single point p, by (E2) there is a connected subspace Y with absolute embedding which meets R and S in subspaces containing respective lines, N and M on point p. Then applying Step 1 to

Y, the transformation $t_Y\colon \mathscr{F}^{(1)}(Y) \to \mathscr{F}^{(2)}(Y)$ exists and is determined up to scalar multiple, and so, taking restrictions of t_Y to N and M and applying (*) with N and M replacing L, we have

1. $t_{Y,N}$ is a scalar multiple of $t_{R,N}$ in $\hom_k(e_1(N), e_2(N))$.
2. $t_{Y,M}$ is a scalar multiple of $t_{S,M}$ in $\hom_k(e_1(M), e_2(M))$.

Since $t_{Y,N}$ and $t_{Y,M}$ agree on $\mathscr{F}^{(1)}(p) = e_1(p)$, we see that a scalar multiple of t_R agrees with t_S on $e_1(p)$, which is $e_1(R \cap S)$ in this case.

Thus, combining the two cases, we have:

(Step 2) If S and R are members of $\mathscr{R}$ forming an edge in Σ, then the two isomorphisms of point–line presheaves $t_R\colon \mathscr{F}^{(1)}(R) \to \mathscr{F}^{(2)}(R)$ and $t_S\colon \mathscr{F}^{(1)}(S) \to \mathscr{F}^{(2)}(S)$ can be adjusted by scalar multiples so that their indexed semilinear transformations agree on all points and lines of $R \cap S$. (In particular, the presheaf isomorphisms t_R and t_S are associated with the same automorphism of k.)

Now suppose that A, B, and C are distinct members of R forming a triangle in the graph $\Sigma = (\mathscr{R}, \sim)$. As we have seen from Step 2, it is possible to adjust the scalars so that the point–line presheaf isomorphisms t_A and t_B agree on the spaces $e_1(p)$ and $e_1(L)$ for the points and lines p and L in $A \cap B$. Similarly, t_A and t_C agree on the $e_1(p)$ and $e_1(L)$ for p and L in $A \cap C$. Now t_B and t_C are determined up to scalar multiple (Step 1) and so agree with each other on any $e_1(p)$ for $p \in A \cap B \cap C$. Thus the agreement of the transformations on the overlapping domains can be arranged if $A \cap B \cap C \neq \varnothing$.

Similarly, if the intersection $A \cap B \cap C$ is empty, by hypothesis there is a connected subspace X meeting each pairwise intersection $A \cap B$, $A \cap C$, and $C \cap B$ nontrivially (see condition (T2) of the main theorem). Clearly, since X is connected and intersects disjoint spaces, it contains a line. Thus, since it has an absolute embedding over k, the point–line presheaves $\mathscr{F}^{(i)}(X)$ obtained by restriction of the embeddings e_i to the subspace X, $i = 1, 2$, are isomorphic by a transformation of presheaves t_X (determined up to scalar multiple). Now by Step 2, since X meets A, B, and C at connected subspaces, the transformations t_A, t_B, and t_C can be adjusted by scalar multiples so as to agree on the e_1-images of points and lines in $A \cap X$, $B \cap X$, and $C \cap X$, respectively. Since X is connected and meets each pairwise intersection among A, B, and C, Step 1 shows that these maps are determined by the scalar multiples used, and the maps t_A, t_B, and t_C must agree between each other on e_1-images of points and lines in these intersections. Thus

(Step 3) If $\{A, B, C\}$ forms a triangle in the graph Σ, the three semilinear transformations t_A, t_B, and t_C can be adjusted by scalar multiplication so that any two or three of them agree on any spaces $e_1(p)$ or $e_1(L)$ in their common domain.

Now suppose that $w = (R_1, R_2, \ldots, R_n)$ is a path in the graph Σ. By Step 2, given the isomorphism t_{R_1}: $\mathscr{F}^{(1)}(R_1) \to \mathscr{F}^{(2)}(R_1)$ of point–line presheaves, each of the isomorphisms t_{R_i}: $\mathscr{F}^{(1)}(R_i) \to \mathscr{F}^{(2)}(R_i)$ can be chosen so that t_{R_i} and $t_{R_{i+1}}$ agree on all spaces $e_i(p)$ and $e_1(L)$ for points and lines p and L in the intersection $R_i \cap R_{i+1}$, $i = 1, 2, \ldots, n-1$. In that sense, the isomorphism of point–line presheaves t_{R_n}: $\mathscr{F}^{(1)}(R_n) \to \mathscr{F}^{(2)}(R_n)$ for the terminal subspace of the path can be thought of as *being determined by the initial isomorphism*: t_{R_1} *through the path* w. Now, by Step 3 it is clear that if path w' is obtained from path w by an elementary $\mathscr{T}$-homotopy in Σ, then the transformation of point–line presheaves of R_n determined by the same t_{R_1} via the path w' is unchanged—it is still t_{R_n}. Clearly the same statement is true if w' is replaced by *any* path $\mathscr{T}$-homotopic to w.

At this point, the hypothesis that Σ is simply connected, shows

(Step 4) Suppose we are given an isomorphism of point–line presheaves t_{R_1}: $\mathscr{F}^{(1)}(R_1) \to \mathscr{F}^{(2)}(R_1)$. Then for each element $S \in \mathscr{R}$, there is a unique isomorphism t_S: $\mathscr{F}^{(1)}(S) \to \mathscr{F}^{(2)}(S)$ such that for any path $p = (R_1, R_2, \ldots, R_n)$ in Σ and any appropriate index i, the presheaf isomorphisms t_{R_i} and $t_{R_{i+1}}$ agree on all terms $e_1(p)$ and $e_1(L)$ for points and lines p and L in $R_i \cap R_{i+1}$.

(Step 5) Suppose that S and R are elements of $\mathscr{R}$ containing a point p. Let t_R and t_S be members of the uniform system of isomorphisms of point–line presheaves as in Step 4. Then the semilinear transformations $t_{R,p}$ and $t_{S,p}$ agree on $e_1(p)$. Similarly if $S \cap R$ contains a line L, the semilinear transformations $t_{R,L}$ and $t_{S,L}$ agree on line L.

In Case 1, the first statement is ensured by the hypothesis (Σ_p) on connectedness of the subgraphs Σ_p. In Case 2, it is automatic from (E2). If the second statement were false, there would exist a line L and subspaces S and R of $\mathscr{R}$ containing L such that the semilinear transformations $t_{R,L}$ and $t_{S,L}$ did not agree on $e_1(L)$. Then there would exist a 1-subspace $e_1(p)$ where these transformations disagree. But that is impossible by the first assertion of this step.

Now we are near the end of the proof. By Step 4, the isomorphisms $\{t_S \mid S \in \mathscr{R}\}$ can be consistently used to define a global isomorphism $\hat{t}$: $\mathscr{F}^{(1)} \to \mathscr{F}^{(2)}$: For each line L one can unambiguously set $\hat{t}_L = t_{S,L}$ for any element S in $\mathscr{R}$ containing L (assumption (2)), that every line lies in a subspace of R (used here for the first time)). By hypothesis such an S

exists, and no matter what S you use, Step 5 assures one that this mapping is well defined. If p is any point, one can set $\hat{t}_p = t_{S,p}$ for any element S of $\mathscr{R}$ which contains point p. This does not depend upon the particular value of S by Step 5. That means there is a uniform isomorphism of point–line presheaves $t: \mathscr{F}^{(1)} \to \mathscr{F}^{(2)}$, as desired. The proof is complete.

4. APPLICATIONS

4.1. *The Geometry* $D_{4,2}$

We begin with a particular geometry studied by Battens and Pasini [BP] —namely the metasymplectic space $D_{4,2}$. Let Δ be the ordinary classical polar space associated with a quadratic form of maximal Witt index on the vector space $V = F^{(8)}$, where F is a field. As is well known, this polar space (of type $O^+(8,F)$) is *oriflame*—that is, the maximal singular subspaces belong to two classes M_1 and M_2 with the property that two maximal singular spaces belong to the same class if and only if their intersection has even codimension in each. In this way, the singular 1-spaces (the polar points), totally singular 2-spaces (the polar lines), and the two classes of maximal singular subspaces M_1 and M_2 comprise a rank four geometry belonging to the diagram D_4.

The *points* of the Lie incidence geometry $D_{4,2}$ are the polar lines. A *line* of this geometry would be the full pencil of polar lines on a polar point and contained in a totally singular 3-space (polar plane) containing the polar point. Thus any point–plane flag of the polar space of type $O^+(8,F)$ defines a line of this $D_{4,2}$-space. (In fact, this space is a special case of a metasymplectic space—in turn, a special type of parapolar space.)

The nodes of this diagram correspond to special subspaces of the $D_{4,2}$ geometry (we refer to Fig. 1). Each $\mathscr{S}_i$, $i = 1,2,3$, is a class of symplecta (convex subspaces which are non-degenerate polar spaces). Each symplecton is of type $O^+(6,F)$—the polar spaces of the Klein quadric.

We wish to lay out carefully the possible relations between symplecta.

$\mathscr{S}_i \times \mathscr{S}_i$. If A and B are two symplecta belonging to the same class, there are three possible relationships: (1) they could be equal; (2) they

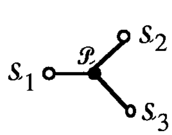


FIG. 1. The diagram of the principle subspaces of the $D_{4,2}$ metasymplectic space.

could intersect at a point; or (3) they could be opposite. Two symplecta A and B are "opposite" if the following is true: $A \cap B = \emptyset$, and there is a bijection $f\colon A \to B$, such that $(a, f(a))$ belong to a common symplecton. If b is collinear but distinguished from $f(a)$ in B, then (a, b) is a *special pair* in the language of parapolar spaces: that is, a and b are at distance two in the point-collinearity graph Δ, but there is a unique point (in this case $f(a)$) collinear with both a and b. Finally, if c is a point of B which is not collinear with $f(a)$, then a and c are at distance three in Δ. The correspondence $f\colon A \to B$ is an isomorphism of polar spaces—that is, x is collinear with y in A if and only if $f(x)$ is collinear with $f(y)$ in B.

$\mathscr{S}_i \times \mathscr{S}_j$: $i \neq j$. Suppose that A and B are symplecta belonging to distinct classes. Then there are just two relations between the symplecta: (1) they intersect at a plane, or (2) they are disjoint, but possess unique planes π_A and π_B in A and B, respectively, such that for some symplecton C, $C \cap A = \pi_A$ and $C \cap B = \pi_B$. Clearly the symplecton C is unique and belongs to the third class $\mathscr{S}_k$ unoccupied by either A or B.

Each projective plane π of the parapolar space $D_{4,2}(F)$ belongs to exactly two symplecta belonging to different families. So there are three types of planes.

Now let $\mathscr{R}$ be the collection of all symplecta of $\Gamma = D_{4,2}(F)$. Thus $\mathscr{R} = \mathscr{S}_1 + \mathscr{S}_2 + \mathscr{S}_3$. Form the graph $\Sigma = (\mathscr{R}, \sim)$, where we write $A \sim B$ if and only if $A \cap B$ is connected and contains a line. In our case, this only happens if $A \cap B$ is a plane and A and B belong to distinct classes.

Now consider a triangle $\{A, B, C\}$ in Σ. It can be shown from first principles below that in this case $A \cap B \cap C$ is a line. (Of course the reader always has the more familiar polar-space model to check assertions about the incidence of flags in this geometry.) Thus Σ turns out to be an interesting graph. (1) It is a tripartite graph with vertex components $\mathscr{S}_i$, $i = 1, 2, 3$; thus every triangle has to spread over all three parts, and (2) it is locally the bipartite incidence graph of points and planes of a $PG(3, D)$ (where, in fact, D is a field).

Now we show that Σ is simply connected. Suppose that $c = (R_0, R_1, \ldots, R_n = R_0)$ is a non-$\mathscr{T}$-contractible pointed circuit in Σ of minimal possible length n. Then c must be *isometrically embedded in* Σ—that is, distances measured in c are the actual distances measured in Σ. Since Σ has diameter three, $n \leq 7$.

Notice that two vertices in Σ are at distance three if and only if they belong to the same class. That makes an isometrically embedded 7-circuit impossible, since the two vertices at distance three from a vertex in the circuit must be adjacent to one another.

If $n = 6$, the circuit has the form $c = (A_1, A_2, A_3, B_1, B_2, B_3, A_1)$ where antipodal pairs A_i and B_i belong to the same class $\mathscr{S}_i$, $i = 1,2,3$. Adjacent symplecta in the circuit intersect at a plane, and if the two planes produced by the intersections of a symplecton with its two neighbors (belonging to distinct classes) would be adjacent with each other, against that fact that c is an isometrically embedded circuit. Thus for any vertex X of the hexagon c, the two planes obtained by intersecting X with its neighbors in c are disjoint planes.

Select a point p in the intersection $A_1 \cap A_2$. Then in A_2, $p^{\perp}$ meets the plane $A_2 \cap A_3$ at a line L. In turn, because of the disjointness of the successive connecting planes, $L^{\perp}$ meets the plane $A_3 \cap B_1$ at a unique point r. Now $p^{\perp} \cap r^{\perp}$ contains the line L, and so the pair $\{p, r\}$ must lie in some symplecton C_1, which clearly belongs to $\mathscr{S}_1$. Thus p and r are corresponding points in the opposite relation between symplecta A_1 and B_1. There is a symplecton $C_3 \in \mathscr{S}_3$ on p. Then C_3 meets A_1, A_2, and C_1 at planes. Now $C_3 \cap C_1$ is a plane on p which meets $r^{\perp}$ at a line N. Then $\pi := \langle N, r\rangle_{C_1}$ is a plane of C_1 belonging to a different class than the plane $C_3 \cap C_1$, and so this plane π lies in a symplecton $C_2 \in \mathscr{S}_2$. Now C_2 is adjacent to A_3 and B_1 since all three contain point r. Also, C_2 is adjacent to C_3 since their intersection contains line N.

The two triangles $\{A_1, A_2, C_3\}$ and $\{A_3, B_1, C_2\}$ are contractible, and so is the 4-circuit $(A_2, C_3, C_2, A_3, A_2)$. Thus c is homotopic to the 6-circuit $c' := (A_1, C_3, C_2, B_1, B_2, B_3, A_1)$. But now the pair of vertices $\{C_3, B_2\}$ are antipodal in c' but are at distance 2 in Σ, since $C_3 \in \mathscr{S}_3$ and $B_2 \in \mathscr{S}_2$; that is, they belong to different classes of symplecta. This means c' decomposes into two contractible pentagons and so itself is contractible, forcing the original c to be contractible, contrary to assumption.

If $n = 5$, one of the classes $\mathscr{S}_i$ has exactly one representative in a minimal non-contractible 5-circuit, so without loss, we may take $c = (A_1, A_2, B_1, B_2, A_3, A_1)$, where again, the subscripts indicate the class from which the symplecton originates. Since c is isometrically embedded, $A_1 \cap B_2 = \varnothing$. Now the planes $A_1 \cap A_2$ and $A_2 \cap B_1$ belong to the same class in A_2 and so meet at a point p. Similarly the planes $A_2 \cap B_1$ and $B_1 \cap B_2$ belong to the same class in B_1 and so meet at a point r. Since $A_1 \cap B_2 = \varnothing$, p and r are distinct points. But they are collinear since they both belong to the "middle" plane $A_2 \cap B_1$. Now the line $L = pr$ lies in a unique element $B_3 \in \mathscr{S}_3$, and one sees that B_3 is adjacent to all of A_1, A_2, B_1, and B_2, making c contractible.

If $n = 4$, the minimal circuit either involves only symplecta from two classes or else symplecta from all three classes. In the former case, $c = A_1$, A_2, B_1, B_2, and A_1 and we can find a B_3 adjacent to all four of these vertices as in the previous paragraph. In the latter case, c has the form

$(A_1, A_2, A_3, B_2, A_1)$. By the same argument on classes of planes, it follows that each of the intersections $A_2 \cap A_1 \cap B_2$ and $A_2 \cap A_3 \cap B_2$ is non-empty. But since A_2 and B_2 can intersect in at most one point, we see that this point is in both A_1 and A_3, and so symplecta meet in a plane and represent a diagonal adjacency in c; thus c is contractible.

This finishes the argument that Σ is a simply-connected graph.

Next we consider triangles $\{A, B, C\}$ in Σ. Clearly each of A, B, and C belongs to a unique class $\mathscr{S}_i$. Suppose $A \cap B \cap C = \emptyset$. Choose a point p in plane $A \cap B$. Then $p^{\perp}$ meets each of the two planes $A \cap C$ and $B \cap C$ at lines, and these lines are disjoint because of the empty intersection of the three symplecta. For the same reason p is not in C, and so, since C is convex, $p^{\perp} \cap C$ is a singular space containing the two aforementioned disjoint lines. But this forces C to be a symplecton of rank at least four, which it is not. Thus all triangles are represented by three symplecta meeting at a common line.

Finally, if we fix a point p of this $D_{4,2}(F)$ geometry, the subgraph Σ_p of Σ induced by the set of symplecta on p is easily seen to be a complete tripartite graph, and so is connected.

Since the polar spaces of type $O^+(6, F)$ all possess an absolutely universal embedding over F, all of the conditions of the main theorem have been met. We conclude that the geometry $D_{4,2}(F)$ possesses an absolute embedding over F.

4.2. *The Geometries $D_{n,n-2}$, $n \geq 5$*

This Lie incidence geometry is indicated by the diagram of Fig. 2, where there are at least five nodes and the dark node indicates the variety of points, $\mathscr{P}$. Flags whose objects are of types corresponding to the nodes neighboring the dark node comprise the variety of lines, $\mathscr{L}$.

The variety $\mathscr{R}$ will be that corresponding to the end node of the long arm of the diagram, as indicated. Thus $\mathscr{R}$ is a class of subspaces of type $D_{n-1,n-3}(F)$ each of which possesses an absolute embedding over F as an induction assumption. This makes sense since we have proved this result for $n = 4$ and here $n \geq 5$.

Now two members of $\mathscr{R}$ either intersect at a subspace of type $D_{n-2,n-4}$ (which is one of the classes of symplecta when $n = 5$) or possess an empty intersection. If three distinct elements of $\mathscr{R}$ pairwise intersect, then all

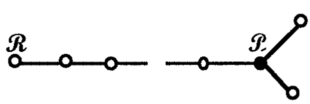


FIG. 2. The diagram for a special polar Grassmannian, $n \geq 5$.

three contain together at least a subspace of type $D_{n-3,n-5}$, for $n > 6$, or a symplecton, or a point if $n = 6$, or 5, respectively. Thus in the graph $\Sigma = (\mathscr{R}, \sim)$, triangles are represented by three subspaces having a non-empty intersection. Finally, for each point p, the subgraph Σ_p is a non-empty clique and so is connected.

Now the graph Σ is easily recognized as the point-collinearity graph of a polar space and so is simply connected. So the main theorem shows that this geometry has an absolute embedding, provided it can be shown that it has any embedding at all. But the latter is provided by some subspace of a $n - 2$-fold wedge product of the natural $2n$-dimensional polar space module with itself.

4.3. *The Grassmannians $A_{n,r}(k)$, Where k Is a Field*

Such a geometry Γ possesses the diagram of Fig. 3, where we assume that $1 < r \leq [(n+1)/2] < n$. Points are the r-spaces of an $n+1$-dimensional vector space V, whose lines are the $(r-1, r+1)$-subspace flags. We assume that $n \geq 4$ since otherwise this is a polar space. The node labelled n corresponds to a class $\mathscr{R}$ of convex subspaces which are themselves Grassmannians of type $A_{n-1,r}$, which by induction are assumed to possess an absolute embedding over the field k. The graph Σ is a complete graph and so are its subgraphs Σ_p. Since $n \geq 4$ and $r \leq [(n+1)/2]$, any three of these Grassmannians of $\mathscr{R}$ have a non-empty intersection. The main theorem now applies, forcing Γ to have an absolute embedding.

4.4. *The Half-Spin Geometries*

Here Γ is the geometry whose points are one class of maximal singular subspaces of an oriflame polar space Δ and whose lines are the subspaces of codimension two in the former. The geometry belongs to the diagram given in Fig. 4 where $n \geq 5$. (When $n = 4$, Γ is a polar space, a case already covered.) Let $\mathscr{R}$ be the class of convex half-spin geometries corresponding to node 1. When $n = 5$, $\mathscr{R}$ is a class of symplecta of rank four, and for $n > 5$, its subspaces are half-spin spaces of rank smaller than n, each possessing an absolute embedding over k by induction. Clearly, the edges (A, B) and triangle (A, B, C) of the graph $\Sigma = (\mathscr{R}, \sim)$ represent cases in which $A \cap B$ and $A \cap B \cap C$ are connected non-empty sub-

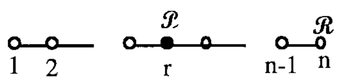


FIG. 3. The diagram for the classical Grassmannians.

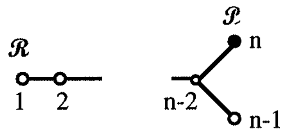


FIG. 4. The diagram for the half-spin geometries.

spaces and the former always contains a line. The graph Σ is the point-collinearity graph of the polar space Δ and so is simply connected. For every half-spin point p, the subgraph Σ_p is a non-empty clique. Finally, the half-spin module provides an embedding for Γ. Thus the main theorem applies to show that Γ has an absolute embedding, a fact first shown by Wells [W].

4.5. *The Polar Grassmannians $D_{n,r}(k)$, Where k Is a Field*

This geometry Γ possesses the diagram in Fig. 5, where we assume that $1 < r \leq n - 3$. Points are the totally singular r-spaces of a quadratic form (V, Q) where V is a vector space of dimension $2n$ and Witt index n with respect to Q. The node indexed by $n - 1$ corresponds to a class of convex subspaces which are Grassmannians of type $A_{n-1,r}$. By the previous section, each of these possesses an absolute embedding over the field k. The graph Σ is the point-collinearity graph of the half-spin geometry and is simply connected by Corollary 5. Any triangle of the graph is represented in the polar space model by three maximal singular subspaces above a common subspace at codimension three. Thus, in this polar Grassmannian Γ, a triangle in Σ represents subspaces of $\mathcal{R}$ whose intersection is non-empty. Finally, the subgraph Σ_p is the point-collinearity graph of a half-spin geometry (if $r < n - 4$) or of a symplecton (if $r = n - 4$ or $n - 3$) and in either case is connected. Thus the main theorem applies once more.

4.6. *Some Dual Polar Spaces*

Here $\Gamma = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ and $\mathcal{L}$ are the maximal and second-maximal singular subspaces of a (non-oriflame) polar space Δ of rank n. Any

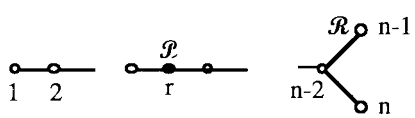


FIG. 5. Numbering of the D_n diagram. Points are the rth node, $1 \leq r \leq n - 3$.

two points at distance 2 lie in a convex generalized quadrangle which we call a *quad*. Of course, as Γ is assumed to be embeddable, the quads must be embeddable. This is a major restriction since many generalized quadrangles (some being duals of classical quadrangles) are not embeddable.

Now we assume that $n \geq 3$. Let $\mathscr{R}$ be the class of convex dual polar subspaces corresponding to the polar points of Δ. If $n = 3$, then $\mathscr{R}$ is $\mathscr{Q}$, the class of quads. By the theorems of Dienst [D] and Tits [T], we may assume that each embeddable thick quad of $\mathscr{R}$ has an absolute embedding over k.

Now two subspaces A and B of $\mathscr{R}$ represent an edge of the graph Σ if and only if they intersect at a subspace corresponding to the second node of the usual C_n diagram—the node corresponding to polar lines of Δ. Thus $A \cap B$ is a line when $n = 3$ and is a dual polar space when $n > 3$. If three such subspaces (A, B, C) form a triangle in the graph Σ, then their intersection is a point if $n = 3$, a line if $n = 4$, or some non-empty dual polar space of rank $n - 3$ if $n > 4$. In any case the intersection $A \cap B \cap C$ is non-empty, which is the important thing.

Now Σ is simply connected since it is the graph of a polar space of rank at least three. Also, for each point $p \in \mathscr{P}$, the subgraph Σ_p is a non-empty clique and so is connected. Now, since Γ is assumed to be embeddable, the main theorem implies that Γ has an absolute embedding.

4.7. *The Spaces $C_{3,2}(k)$, Where k Is a Field*

Let Δ be a classical polar space of rank 3 which has at least three planes on each line (i.e., it is not oriflame) defined by a symplectic, Hermitian, or quadratic form on a vector space V over a field. The geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ has the polar lines of Δ as its set of points, $\mathscr{P}$, and the point–plane line pencils of Δ as the collection $\mathscr{L}$ of lines. The point-collinearity graph of Γ has diameter three.

The symplecta of Γ are a class $\mathscr{R}$ of generalized quadrangles isomorphic to the embeddable residual quadrangles of Δ. Since k is a field, one can find an embedding $e: \Gamma \to \mathbf{P}(W)$ where W is a subspace of $V \wedge V$.

For any two distinct quads A and B, either $A \cap B$ consists of a single point p or else $A \cap B = \varnothing$ and there is a bijection $f: A \to B$, such that corresponding pairs $(a, f(a)) \in A \times B$ lie in a quad. If b is a point of $B - \{f(a)\}$ collinear with $f(a)$, then (a, b) is a "special pair"—that is, $|a^\perp \cap b^\perp| = 1$—and if $b \in B - f(a)^\perp$, then $d(a, b) = 3$.

Now form the graph $\Sigma = (\mathscr{R}, \sim)$ where $A \sim B$ if and only if $A \cap B = \{p\}$, a single point. In this case, there exists a projective plane π in Γ containing p and intersecting each of the quads at a line on p. Thus we have the condition (E2) of the main theorem on edges.

Also, if (A, B, C) is a triangle in the graph $\Sigma = (\mathscr{R}, \sim)$, then either $A \cap B \cap C = \emptyset$ or it consists of a single point. In either case, there exists a projective plane X meeting each of the three quads at lines, and in the case that $A \cap B \cap C$ is empty, this plane contains the three points of the pairwise intersections of the quads.

Now all symplecta here are embeddable, and the graph Σ is the point-collinearity graph of the polar space Δ of rank three, and so is simply connected. Finally, the subgraph Σ_p is easily seen to be a non-empty clique corresponding to the points of a polar line. The main theorem now shows that as Γ is embeddable, it has an absolute embedding.

(Note that this is the first occasion for which it was necessary to invoke Case 2 of the definition of Σ and the condition (E2). This circumstance will occur again for the geometries $C_{n,2}$ and $E_{8,2}$.)

4.8. *The Polar Grassmannians* $C_{n,r}(k)$, $1 < r < n$, $n \geq 4$, *Where k Is a Field*

The points of Γ are totally singular or isotropic r-subspaces, $1 < r < n$, of a vector space V with respect to a symplectic/Hermitian or quadratic form, forming a classical polar space Δ. A line of Γ is any $(r-1, r+1)$-flag of Δ. Then Γ is a Lie incidence geometry which belongs to the diagram of Fig. 6. Then as $n \geq 4$, Γ contains two classes of symplecta. There is first a class $\mathscr{S}$ corresponding to the node labelled $r-1$, whose members have rank $n - r + 1$. Then there is a class of symplecta of type $O^+(6, k)$ corresponding to flags of type $(r-2, r+2)$ (or just type 4 if $r = 2$). We designate $\mathscr{R}$ to be the collection of all convex polar Grassmannians which correspond to the node labelled "1." If $r = 2$, then $\mathscr{R} = \mathscr{S}$.

Now form the graph $\Sigma = (\mathscr{R}, \sim)$ where we write $A \sim B$ if and only if A and B are distinct members of $\mathscr{R}$ which have a non-empty intersection. Note that if $r = 2$, such an intersection $A \cap B$ is a single point (as in Case 2 of the definition of Σ in the main theorem). On the other hand if $r = 3$, $A \cap B$ is a symplecton of rank two if $r = n - 1$ or higher rank if $r < n - 1$, while if r is greater than 3, the intersection is itself a polar Grassmannian (as in Case 1 of the main theorem). Note that in the cases that $r \geq 3$, $A \cap B$ is connected and contains a line (condition (E1)).

In the case $r = 2$, when $A \cap B = \{p\}$ and (A, B) is an edge of Σ, there always exists a projective plane Y (an object of type 3) which contains p and meets both A and B at lines. Thus in this case, we have condition (E2) operating in Case 2 of the definition of Σ in the main theorem.

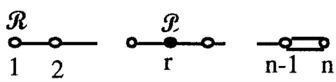


FIG. 6. The diagram for the polar Grassmannians. Here $1 < r < n$.

A triangle (A, B, C) of the graph Σ represents three elements of $\mathscr{R}$ incident with at least one common object of type 3. If $r = 2$, that object is a plane X meeting each of A, B, and C at lines and containing the pairwise intersections $A \cap B$, $A \cap C$, and $B \cap C$. If $r > 2$, this object of type three possesses points in all three of the spaces so $A \cap B \cap C$ is non-empty.

Now as noted, if $r = 2$, $\mathscr{R}$ is a class of embeddable symplecta. Otherwise $\mathscr{R}$ is a class of polar Grassmannians of type $C_{n-1,r}(k)$ where $1 < r < n - 1$ and these have an absolute embedding by induction on n or by the result on $C_{3,2}(k)$ proved in the previous section.

Now Σ is the simply-connected point-collinearity graph of the polar space Δ and for each point p of $\mathscr{P}$, Σ_p is a non-empty clique. The main theorem now applies.

4.9. *Classical Metasymplectic Spaces over a Field*

Here, Γ belongs to the diagram of Fig. 7, where $\mathscr{P}$ and $\mathscr{L}$ are the points and lines, Π is a class of projective planes, and $\mathscr{S}$ is a class of symplecta of rank 3. Γ is a parapolar space. Any three pairwise collinear points lie together in some projective plane. There are a number of possibilities for a pair of symplecta $(A, B) \in \mathscr{S} \times \mathscr{S}$: (i) $A = B$; (ii) $A \cap B$ is a projective plane; (iii) $A \cap B$ is a single point; (iv) $A \cap B = \varnothing$, but there is a unique symplecton C meeting both A and B at planes; and (v) A and B are opposite, which means that $A \cap B = \varnothing$ and for each point $a \in A$, there is a unique point $f(a) \in B$ such that $(a, f(a))$ lies in a symplecton and the mapping f: $A \to B$ is an isomorphism. The reader should recognize all of these relations from the previously studied $D_{4,2}(k)$, another metasymplectic space.

We set $\mathscr{R} = \mathscr{S}$ and form the graph $\Sigma = (\mathscr{R}, \sim)$ where (A, B) is an edge of Σ if and only if $A \cap B$ is a plane. Then for any triangle (A, B, C) of Σ, $A \cap B \cap C$ is a plane or a line (this is just the gamma space property of the dual metasymplectic space $(\mathscr{S}, \Pi)$).

Now Σ is the point-collinearity graph of a classical metasymplectic space of type $F_{4,1}$ and so is simply connected by Corollary 5. Finally its subgraph Σ_p is the collinearity graph of a polar space and so is connected. Thus all the conditions of the main theorem hold, so Γ has an absolute embedding, if it is embeddable at all. (Note that when Γ is the metasymplectic space

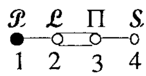


FIG. 7. The diagram for the metasymplectic spaces.

obtained from twisted E_6, or from E_7, it may not be embeddable since its planes, though Moufang, may not be Desarguesian.)

4.10. *A Geometry of Lines Derived from $F_{4,2}(k)$, Where k Is a Field*

We are concerned here with a geometry Γ whose points $\mathscr{P}$ are the lines of a classical metasymplectic space Δ and whose lines $\mathscr{L}$ are the line-pencils defined by point−plane flags of Δ. Then Γ belongs to the diagram of Fig. 8. The objects of type 4 form a class $\mathscr{R}$ of embeddable convex subspaces of type $C_{3,2}(k)$, which, as we have already seen, possess an absolute embedding. We define the graph $\Sigma = (\mathscr{R}, \sim)$ by asserting that two members of $\mathscr{R}$ are adjacent if and only if they intersect at a plane (object of type 3). If (A, B, C) is a triangle of Σ, then $A \cap B \cap C$ is a single point. The graph Σ is again the simply-connected point-collinearity graph of $F_{4,1}(k)$. The subgraph Σ_p is a clique. Thus by the main theorem, the geometry Γ has an absolute embedding, if it has an embedding at all.

4.11. *Lie Incidence Geometries of Exceptional Type*

In each of the cases below, after the display of indices for the nodes of the relevant Dynkin diagram we tabulate the relevant data for each geometry in a row. The first column lists the name of the Lie incidence geometry, the second column under "$\mathscr{R}$" lists the node corresponding to the elements of $\mathscr{R}$. Σ is a point-collinearity graph of the geometry listed in the third column. The subspace $A \cap B$ for an edge (A, B) of Σ is listed in column four. If it is a point, the phrase "(E2)" indicates that there is a projective plane containing the pairwise intersection of the spaces. Column five lists the isomorphism types of $A \cap B \cap C$ for a triangle (A, B, C) of Σ. The symbol "(T2)" indicates that condition (T2) is verified for a projective plane when $A \cap B \cap C = \emptyset$. In column six the nature of the subgraph Σ_p is indicated. If it is a clique, that is indicated. If it is not a clique, it is the point-collinearity graph of the indicated geometry.

The following facts are intended to be displayed by the table:

1. Σ is a simply connected graph listed in Corollary 5 (column two).

2. The subspaces in $\mathscr{R}$ are each known to have an absolute embedding by a previous case (column three).

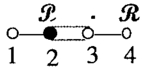


FIG. 8. The diagram of the Lie incidence geometry of type $F_{4,2}$.

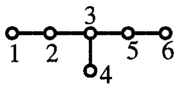
3. Σ is defined by Case 1, and $A \cap B$ is connected and contains a line or Σ is defined as in Case 2 and (E2) holds (column four).

4. $A \cap B \cap C$ is non-empty or (T2) holds (column five).

5. The subgraph Σ_p must be connected (column six).

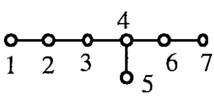
These six conditions being verified, the main theorem implies that the Lie incidence geometry of the first column possesses an absolute embedding if it is embeddable.

4.11.1. Lie incidence geometries derived from the building of type $E_6(k)$, where k is a field. The diagram numbering used here is



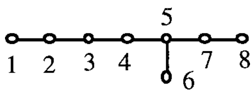
		E_6			
Geometry	$\mathscr{R}, \Sigma$	R	$A \cap B$	$A \cap B \cap C$	Σ_p
$E_{6,1}$	6, $E_{6,1}$	$D_{5,1}$ symp	$PG(4)$	Plane	Polar space $D_{5,1}$
$E_{6,2}$	6, $E_{6,1}$	$D_{5,2}$	$A_{4,2}$	Plane	Clique
$E_{6,3}$	6, $E_{6,1}$	$D_{5,3}$	$A_{4,2}$	Point	Plane
$E_{6,4}$	6, $E_{6,1}$	$D_{5,5}$	$PG(4)$	Line	Clique

4.11.2. Lie incidence geometries derived from the building of type $E_7(k)$. The diagram numbering is



		E_7			
Geometry	$\mathscr{R}, \Sigma$	R	$A \cap B$	$A \cap B \cap C$	Σ_p
$E_{7,1}$	7, $E_{7,7}$	$D_{6,1}$ symp	$PG(5)$	$PG(3)$	$E_{6,1}$
$E_{7,2}$	7, $E_{7,7}$	$D_{6,2}$	$A_{5,2}$	$A_{3,2}$-symp	$D_{5,1}$
$E_{7,3}$	7, $E_{7,7}$	$D_{6,3}$	$A_{5,3}$	$PG(3)$	$PG(4)$
$E_{7,4}$	7, $E_{7,7}$	$D_{6,4}$	$A_{5,4}$	Point	Plane
$E_{7,5}$	7, $E_{7,7}$	$D_{6,5}$ half-spin	$PG(5)$	Line	$PG(6)$
$E_{7,6}$	1, $E_{7,1}$	$E_{6,2}$	$D_{5,2}$	$A_{4,2}$	$PG(5)$
$E_{7,7}$	1, $E_{7,1}$	$E_{6,1}$	$D_{5,1}$ symp	$PG(4)$	$D_{6,1}$

4.11.3. Lie incidence geometries derived from the building of type $E_8(k)$. The diagram is numbered in this way:



		E_7			
Geometry	$\mathscr{R}, \Sigma$	R	$A \cap B$	$A \cap B \cap C$	Σ_p
$E_{8,2}$	1, $E_{8,1}$	$E_{7,1}$	Point (E2)	$\varnothing$ (T2)	Line
$E_{8,3}$	1, $E_{8,1}$	$E_{7,2}$	$E_{6,1}$	Point	Plane
$E_{8,4}$	1, $E_{8,1}$	$E_{7,3}$	$E_{6,2}$	$D_{5,5}$	$PG(3)$
$E_{8,5}$	1, $E_{8,1}$	$E_{7,4}$	$E_{6,3}$	$D_{5,3}$	$PG(4)$
$E_{8,6}$	1, $E_{8,1}$	$E_{7,5}$	$E_{6,4}$	$D_{5,4}$	$PG(7)$
$E_{8,7}$	1, $E_{8,1}$	$E_{7,6}$	$E_{6,5}$	$D_{5,2}$	$PG(6)$
$E_{8,8}$	1, $E_{8,1}$	$E_{7,7}$	$E_{6,6}$	$D_{5,1}$	$D_{7,1}$
$E_{8,1}$	8, $E_{8,8}$	$E_{7,1}$	$PG(6)$	$PG(4)$	$E_{7,7}$

Remark. Note that many of the other cases above the last line of the table could have been handled by letting $\mathscr{R}$ be the subspaces of type 8. For example, the exceptions for $E_{8,2}$ in the first line of the table could be avoided and the first row of the table could have read:

$E_{8,2}$	8, $E_{8,8}$	$E_{7,2}$	$A_{6,2}$	$A_{4,2}$	$E_{6,1}$

ACKNOWLEDGMENTS

The authors express their gratitude to Professor Antonio Pasini who pointed out to them how his theory of plate complexes can address questions of simple-connectedness of graphs. His Theorem 12.64 embodies most of this versatile theory which can show that virtually any graph defined by relations among flags of a building geometry is $\mathscr{C}$-simply connected for an appropriate class of circuits $\mathscr{C}$—and in very many cases $\mathscr{C}$ includes all triangles.

The authors also acknowledge their gratitude to the referee who eliminated several misstatements and greatly enhanced the readibility of the text in other ways.

REFERENCES

[A] M. Aschbacher, "Sporadic Groups," Cambridge Tracts in Math, No. 104, Cambridge Univ. Press, Cambridge, UK, 1994.

[AS] M. Aschbacher and Y. Segev, Extending morphisms of groups and graphs, *Ann. of Math.* **135** (1992), 297–323.

[BP] N. Battens and A. Pasini, On certain hyperplanes and projective embeddings of D_4-buildings, *Internat. J. Game Theory Algebra* **8** (1998), 137–166.

[D] K. J. Dienst, Verallgemeine Vierecke in Projectiven Raumen, *Arch. Math.* (*Basel*) **45** (1980), 177–186.

[K] A. Kasikova, "Hyperplane Complements of Lie Incidence Geometries," Ph.D. dissertation, Kansas State University, 1999.

[P] A. Pasini, "Diagram Geometries," Oxford Univ. Press, New York, 1994.

[R] M. Ronan, Embeddings and hyperplanes of discrete geometries, *European J. Combin.* **8** (1987), 179–185.

[S] E. Shult, Embeddings and hyperplanes of Lie incidence geometries, *in* "Groups of Lie Type and Their Geometries" (L. Martino *et al.*, Eds.), pp. 215–232, London Mathematical Society Lecture Notes, No. 207, Cambridge Univ. Press, Cambridge, UK, 1995.

[S1] E. Shult, "Coverings of Graphs," Lecture Notes, Kansas State University, 1997.

[T] J. Tits, "Buildings of Spherical Type and Finite BN-Pairs," Lecture Notes in Mathematics, No. 389, Springer-Verlag, Berlin/New York, 1974.

[V] F. Veldkamp, Polar geometry, I–IV, *Indag. Math.* **21** (1959), 512–551.

[Wt] G. M. Weetman, A construction of locally homogeneous graphs, *J. London Math. Soc.* (*2*) **50** (1994), 68–86.

[W] A. Wells, Universal projective embeddings of Grassmann, half-spinor and dual-orthogonal geometries, *Quart. J. Math. Oxford* **34** (1983), 375–386.